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Some cosmological solutions of a scalar–tensor theory of gravitation

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Abstract. Some cosmological solutions of the scalar–tensor theory of gravitation are derived. The results differ from Friedmann and Brans–Dicke cosmologies, though not sufficiently to enable an immediate decision between these theories to be made. In the case of the solar system the new theory is a special case of theories discussed by Bergmann and Wagoner.

1. Introduction

In a previous work (Bicknell and Klotz 1976) we considered the derivation of the relativistic field equation of a conformally invariant scalar–tensor theory of gravitation with a long-range scalar field ϕ. If we write

$$\chi = \phi + 12\mu\nu, \tag{1}$$

where μ and ν are constants, these may be written in the form

$$(\mu^2 - \tfrac{1}{12}\chi^2)G_{ij} + \tfrac{1}{3}\chi_{,i}\chi_{,j} - \tfrac{1}{12}\chi_{,k}\chi^{,k}g_{ij} + \tfrac{1}{6}\chi(g_{ij}\chi^{,k}{}_{;k} - \chi_{;ij}) - (\phi^4/8q)g_{ij} = -8\pi\gamma^2 T_{ij} \tag{2}$$

and

$$\Box\phi + \tfrac{1}{6}R\phi - \phi^3 = -(16\pi\nu\gamma^2/\mu)T. \tag{3}$$

Here

$$\chi_{,i} = \phi_{,i}, \qquad q = 1 - 12\nu^2,$$

γ^2 is a constant related to the gravitational constant, G_{ij} is the Einstein tensor, T_{ij}, the energy–stress–momentum tensor, R, the Ricci invariant and $T = g^{ij}T_{ij}$. Latin indices go from 1 to 4 and the summation convention is obeyed. We may note that the sign (*a priori* arbitrary) of the ϕ^4 and ϕ^3 terms is chosen so that a perturbation on a constant solution should give waves travelling at a speed less than that of light. In this paper, we shall consider some cosmological solutions of the equations (2) and (3) under several simplifying assumptions. First we shall assume that the metric is a Robertson–Walker metric of the form

$$ds^2 = a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\psi^2) - dt^2\right) \tag{4}$$

where the factor a is a function of the coordinate time only and k is ±1 or 0. When

$k=+1$ this metric is thus conformally related to an Einstein universe for the particular coordinate system chosen. Since all points on the hypersurface $t=$ constant of the Robertson–Walker world are equivalent, it follows that ϕ (and χ) must likewise be a function of t only. We assume also that under a conformal mapping

$$g_{ij}=e^{\sigma}g'_{ij}, \qquad (\phi=e^{-\sigma/2}\phi'), \tag{5}$$

χ transforms also according to

$$\chi=e^{-\sigma/2}\chi'. \tag{6}$$

It has been shown in Bicknell and Klotz (1976) that the above equations are the field equations of a conformally invariant theory expressed in a particular gauge, that in which stress energy is conserved. As they stand they do not appear to be manifestly conformally invariant. However, under the transformations (5) and (6) the only conformal-invariance-breaking term is the first term μ^2G_{ij} and we can make use of this fact in calculating the left-hand side of the field equations in the above metric. The conformal transformation

$$e^{\sigma}=a^2 \tag{7}$$

maps the Robertson–Walker world onto a quasi-Einstein one, for which the calculation of G_{ij}, etc is much simpler. We have in fact

$$G_1^{1'}=G_2^{2'}=G_3^{3'}=-k, \qquad G_4^{4'}=-3k, \qquad R'=6k. \tag{8}$$

Finally, we assume a pressure-less universe with

$$T^{ij}=\rho c^2u^iu^j \tag{9}$$

(re-introducing the speed of light, c) and with matter moving along t-lines, so that

$$u^i=(0,0,0,\mathrm{d}t/\mathrm{d}s)=(0,0,0,1/a) \tag{10}$$

since along these paths

$$\mathrm{d}s=a\,\mathrm{d}t. \tag{11}$$

2. The equations determining ϕ' and a

In our particular gauge the energy–stress–momentum tensor is conserved:

$$T^{ij}{}_{;j}=0. \tag{12}$$

The assumptions made above concerning the global behaviour of matter imply that the only non-zero component of T_{ij} is

$$T_{44}=\rho c^2a^2, \tag{13}$$

so that $T=\rho c^2$ and equations (12) give (Weinberg 1972, p 472)

$$\rho c^2a^3=A, \tag{14}$$

a constant which we take to be positive together with positive a. A straightforward calculation now shows that the field equations determining ϕ' and a, are

$$k\left(\mu^2-\frac{1}{12}\frac{(\phi'+12\mu\nu a)^2}{a^2}\right)+\mu^2\left(\frac{2\ddot{a}}{a}-\frac{\dot{a}^2}{a^2}\right)+\frac{1}{12a^2}(\dot{\phi}'+12\mu\nu\dot{a})^2$$
$$-\frac{\phi'+12\mu\nu a}{6a^2}(\ddot{\phi}'+12\mu\nu\ddot{a})-\frac{\phi'^4}{8qa}=0 \qquad (15)$$

$$k\left(\mu^2-\frac{1}{12}\frac{(\phi'+12\mu\nu a)^2}{a^2}\right)+\mu^2\frac{\dot{a}^2}{a^2}-\frac{(\dot{\phi}'+12\mu\nu\dot{a})^2}{12a^2}-\frac{\phi'^4}{24a^2q}-\frac{8\pi\gamma^2}{3}(\rho c^2a^2)=0, \qquad (16)$$

and

$$\ddot{\phi}'+k\phi'+\phi'^3=-(16\pi\nu\gamma^2/\mu)A. \qquad (17)$$

Also, contraction of (2) with g^{ij} gives

$$\ddot{a}+ka+\frac{\nu}{\mu q}\phi'^3=4\frac{\pi\gamma^2}{3\mu^2}A. \qquad (18)$$

The dots above denote differentiation with respect to t. Eliminating ϕ'^3 between equation (17) and (18), we get

$$\frac{d^2}{dt^2}\left(a-\frac{\nu\phi'}{\mu q}\right)+k\left(a-\frac{\nu\phi'}{\mu q}\right)=\frac{4\pi\gamma^2A}{3\mu^2q}. \qquad (19)$$

We must now determine the initial conditions under which the above equations are to be solved. Let us suppose (the 'big bang' hypothesis) that the universe is at present evolving from an highly condensed initial state. Then, to a good approximation, we may take

$$a(0)=0, \qquad (20)$$

and because (5) and (7), also

$$\phi(0)=0. \qquad (21)$$

The functions ϕ' and a are continuous and at least twice differentiable and, if the initial scalar field (ϕ'/a) is finite,

$$\lim_{t\to 0}\frac{\phi'}{a}=\lim_{t\to 0}\frac{\dot{\phi}'}{\dot{a}}=\lim_{t\to 0}\frac{\ddot{\phi}'}{\ddot{a}}=\frac{-(16\pi\nu\gamma^2/\mu)A}{4\pi\gamma^2/3\mu^2}=-12\mu\nu$$

from equations (17) and (18).

Equation (16) can be written in the form

$$k(a^2\mu^2-\tfrac{1}{12}(\phi'+12\mu\nu a)^2)+\dot{a}^2\{\mu^2-\tfrac{1}{12}[(\dot{\phi}'/\dot{a})+12\mu\nu]^2\}-(1/24q)\phi'^4$$
$$-(8\pi\gamma^2A/3)a=0. \qquad (22)$$

On taking the limit as $t\to 0$, we get

$$\dot{a}(0)=0, \qquad (23)$$

because of (22) and since $\mu\neq 0$, equation (22) now implies also that

$$\dot{\phi}'(0)=0, \qquad (24)$$

and so (20), (21), (23) and (24) constitute a complete set of the initial conditions for equations (15), (16), (17) and (19). We may observe that compatibility of the latter set of equations is ensured by their derivation from a variational principle demonstrated elsewhere (Bicknell and Klotz 1976).

Equations (19) now give

$$a-\frac{\nu\phi'}{\mu q}=\begin{cases}\dfrac{2\pi\gamma^2 A}{3\mu^2 q}t^2 & k=0\\[2ex] \dfrac{4\pi\gamma^2 A}{3\mu^2 q}(1-\cos t) & k=+1\\[2ex] \dfrac{4\pi\gamma^2 A}{3\mu^2 q}(\cosh t-1) & k=-1.\end{cases} \tag{25}$$

As equation (17) shows, the scalar field ϕ' is an elliptic function. Once ϕ' is known it may be substituted into the above equations to give an expression for $a(t)$. However, the resulting expressions for the Hubble constant, deceleration parameter, proper time since $t=0$, etc are all rather unwieldy and do not yield much information. Consequently in the following section we present some results of numerical calculations of the relevant cosmological parameters.

3. Observational parameters and numerical results

Two of the most important parameters in cosmological models are the Hubble parameter

$$H=\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\tau}, \tag{26}$$

and the deceleration parameter

$$\alpha=-\frac{a\,\mathrm{d}^2a/\mathrm{d}\tau^2}{(\mathrm{d}a/\mathrm{d}\tau)^2}, \tag{27}$$

where the proper time τ since the initial singularity is given by

$$\tau=\frac{1}{c}\int_0^t a(t)\,\mathrm{d}t. \tag{28}$$

In terms of the coordinate time therefore

$$H=c(\dot{a}/a^2) \qquad \text{and} \qquad 1-\alpha=a\ddot{a}/\dot{a}^2.$$

From equations (18) and (14) we now get

$$(\alpha-1)\frac{H^2}{c^2}=\frac{k}{a^2}-\frac{4\pi\gamma^2}{3\mu^2}\rho c^2+\frac{\nu\phi'^3}{\mu q a^3}. \tag{29}$$

Comparison with the corresponding expression in Einstein's theory now gives a cosmological 'constant'

$$\Lambda=-\frac{3\nu}{2\mu q}\frac{\phi'^3}{a^3}. \tag{30}$$

It was found that when $\nu \simeq 0{\cdot}1$ and $\mu \simeq 10^{-26}\ \mathrm{m}^{-1}$ the calculated values of the above parameters were different from the corresponding Friedmann values. Several numerical integrations were performed in the neighbourhood of these values, and several values of A were taken corresponding to different values of the deceleration parameter. (In the Friedmann models A is related to the deceleration parameter α and the Hubble parameter H by

$$A = \alpha \left| \frac{k}{2\alpha - 1} \right|^{3/2} \frac{3c^5}{4\pi GH}.)$$

The numerical integration of the field equations was continued until a value of $H = 55\ \mathrm{km\ s^{-1}\ Mpc^{-1}}$ was obtained, this being one of the most recent determinations of the Hubble constant (Gott *et al* 1974). The value of γ^2/μ^2 was chosen in order that the value of G, given by equation (35) in the last section of this paper, should obtain its present value at the value of H given previously.

In table 1 we present a sample of these calculations. The columns headed α_{F} and τ_{F} give respectively the Friedmann values of the deceleration parameter and of the proper time (since the 'big bang') corresponding to the above Hubble constant and the computed density of the scalar–tensor model. The third column in table 1, $-\chi/\mu$, refers to the background value of $-(\phi + 12\mu\nu)/\mu$ which determines solar system predictions. These will be discussed in the last section of this paper. In the last column the value of $(1/G)\ \mathrm{d}G/\mathrm{d}\tau$ is also given. For all models $\nu = 0{\cdot}1$.

4. Discussion of the results

The values of the deceleration parameter and proper time do not reveal any large differences from the corresponding Friedmann values. Perhaps the most sensitive cosmological test of the present theory is provided by $(1/G)\ \mathrm{d}G/\mathrm{d}\tau$ although our values of this quantity are at least two orders of magnitude less than the observational limit

$$\left| \frac{1}{G} \frac{\mathrm{d}G}{\mathrm{d}\tau} \right| < 4 \times 10^{-10}\ \mathrm{yr}^{-1}. \tag{31}$$

A point of interest is that in our theory $(1/G)\ \mathrm{d}G/\mathrm{d}\tau$ is positive whereas in the Brans–Dicke theory it is negative.

Unfortunately, in order to determine $\mu = 10^{-26}$ and $\nu = -10^{-1}$, for example, α and ρ would have to be known to at least two significant figures. Thus, although the calculated trends in the values of cosmological parameters distinguish our theory from other cosmologies, Friedmann or Brans–Dicke, it is impossible to decide between their objective validity on the basis of the known observational data. On the other hand, the scalar–tensor theory presented here suggests certain tests based on observations within the solar system.

5. Solar system tests

In considering solar system tests of the scalar–tensor theory we may neglect the term ϕ^4 in the Lagrangian (Bicknell and Klotz 1976). The action integral then becomes

$$I = \int \left[\frac{\mu^2}{\gamma^2} \left(1 - \frac{1}{12} \frac{\chi^2}{\mu^2} \right) R + \frac{\mu^2}{2\gamma^2} \left(\frac{\chi}{\mu} \right)_{,i} \left(\frac{\chi}{\mu} \right)^{,i} - 8\pi L_{\mathrm{m}} \right] \sqrt{-g}\, \mathrm{d}^4 x \tag{32}$$

Table 1. Calculated values of the various parameters.

μ (m^{-1})	γ^2/μ^2 ($\times 10^{-45}$ SI units)	$-\chi/\mu$	Λ (m^{-2})	ρ (10^{-28} kg m^{-3})	α	α_F	τ (10^9 yr)	τ_F (10^9 yr)	$(1/G)\,dg/d\tau$ (yr^{-1})
$k=-1$, $A=5\times10^{68}$ SI units									
10^{-26}	8·091	0·414	$8{\cdot}27\times10^{-54}$	7·52	0·03	0·07	16·1	15·7	$3{\cdot}03\times10^{-12}$
$k=-1$, $A=5\times10^{69}$ SI units									
10^{-26}	8·141	0·361	$1{\cdot}01\times10^{-53}$	2·73	0·19	0·24	13·8	13·5	$2{\cdot}91\times10^{-12}$
$k=0$, $A=10^{69}$ SI units									
10^{-26}	8·159	0·323	$1{\cdot}15\times10^{-53}$	48·1	0·36	0·42	12·5	12·3	$2{\cdot}73\times10^{-12}$
$k=+1$, $A=5\times10^{69}$ SI units									
10^{-26}	8·197	0·267	$1{\cdot}39\times10^{-53}$	97·0	0·77	0·84	10·8	10·6	$2{\cdot}35\times10^{-12}$

and is formally equivalent to a special case of the Bergmann–Wagoner theories (Bergmann 1968, Wagoner 1970) for which their scalar field ψ is given by

$$\psi = \frac{\mu^2}{\gamma^2}\left(1 - \frac{1}{12}\frac{\chi^2}{\mu^2}\right), \tag{33}$$

and their arbitrary function ω (of ψ), by

$$\omega = \frac{3\gamma^2}{2\mu^2}\frac{\psi}{1-\gamma^2/\mu^2\psi}. \tag{34}$$

If $\psi = \psi_0$ far outside the solar system, the parameters which determine the deflection of light, time dilatation and the precession of the planetary perihelion are

$$\omega = \omega(\psi_0),$$

itself, and

$$\beta = (2\omega + 4)^{-1}(2\omega + 3)^{-2}\left.\frac{d\omega}{d\phi}\right|_{\psi=\psi_0}.$$

Incidentally, as $\omega \to \infty$ and $\beta \to 0$ the theory becomes equivalent to Einstein's.

The gravitational constant at the present epoch, as determined by Ni (1972) is

$$\frac{G_0}{c^4} = \frac{\gamma^2}{\mu^2}\frac{1+\frac{1}{36}\chi_0^2/\mu^2}{1-\frac{1}{12}\chi_0^2/\mu^2}, \tag{35}$$

so that in the natural units in which $G_0 = c = 1$,

$$\frac{\gamma^2}{\mu^2} = \frac{1-\frac{1}{12}\chi_0^2/\mu^2}{1+\frac{1}{36}\chi_0^2/\mu^2},$$

whence

$$\beta = \frac{1}{216}\frac{\chi_0^2}{\mu^2}\frac{1-\frac{1}{12}\chi_0^2/\mu^2}{1+\frac{1}{36}\chi_0^2/\mu^2}. \tag{36}$$

The observational constraints placed on ω and β (Will 1972) are

$$\omega > 6 \qquad \text{and} \qquad -0{\cdot}46 < \beta < 0{\cdot}64.$$

Hence we get

$$|\chi_0/\mu| < 1{\cdot}55, \tag{37}$$

as an upper limit on the background field. Note that the values of $|\chi_0/\mu|$ in table 1 are all consistent with this constraint.

6. Conclusions

Although the wave equation for the scalar field ϕ is only a seemingly slight modification of that of the Brans–Dicke theory the characters of the two theories differ considerably. In our theory the effect of the scalar field on the gravitational constant is not as marked. There is also a strong interplay between cosmology and solar system observations. The background (cosmological) value of the scalar field determines the deviation of the precession of perihelion and bending of light from their Einstein values.

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